Section 9-1: Exponential Functions

Objective: To write exponential functions
To solve exponential equations

Review: What did you learn yesterday about the equation $y = a(b)^x$

Writing Exponential Equations

If you are given the initial value and one other point, you can write an exponential function.

Example: Find an exponential function that passes through the points (0, 8) and (3, 4)

Example: In the year 2000, the population of a small town was 12,000 people. In 2005, the population had increased to 15,000 people. Find an exponential equation to model the growth of the town.

Find the projected population of the town in 2010.

Solving Exponential Equations

Exponential equations are equations in which a variable occurs as an exponent. The Property of Equality is often useful in solving exponential equations:

If b is a positive number other than 1, then $b^x = b^y$ if and only if x = y.

Example: Solve $2^x = 2^8$

Example: Solve $6^{3y} = 6^{5y-4}$

Sometimes, when equations have different bases, they can be rewritten with a common base. It will be helpful to review the properties of exponents you learned in Lesson 6-1.

Product of Powers :

Quotient of Powers:

Negative Exponents:

Power of a Power:

Example: Solve the following equations by finding a common base

$$2^4 = 8^{x-1}$$

$$9^{3x} = 27^{x-2}$$

$$\left(\frac{1}{25}\right)^{2x} = 5^{x-5}$$

Lesson 9-2: Logarithms

Objectives: To rewrite exponential equations as logarithms
To evaluate simple logarithmic expressions

Warm-Up: Solve each equation by finding the common base

$$5^{x-3} = \frac{1}{25} \qquad\qquad \left(\frac{1}{7}\right)^{y-3} = 343$$

It is not always possible to solve exponential equations by finding a common base. For example, try finding x for these equations:

$$10^x = 100 \qquad 10^x = 500 \qquad 10^x = 1000$$

Logarithms are used to solve this type of exponential equation. A logarithm is simply the answer to the question, “What power does the base have to be raised to in order to get the desired number?”

Use your calculator to find the following: (Note that on your calculator, the “log” key means “$\log_{10}$” or “log base 10”)

$$x = \log_{10} 100 \qquad x = \log_{10} 500 \qquad x = \log_{10} 1000$$

Logarithms, or logs, are simply exponents. The logarithm of y with base b is defined as

$$\log_b y = x \text{ if and only if } b^x = y$$

Note that in both equations, b is called the base and must be positive.

You know that every division equation can be written as a multiplication problem. Similarly, every log equation can be written as an exponential equation and every exponential equation can be written as a log statement.

Example: Write the logarithms in exponential form.

$\log_5 25 = 2$ $\log_4 256 = 4$ $\log_{10} \frac{1}{100} = -2$ $\log_p q = r$

Example: Write the exponential equations in logarithmic form.

$5^3 = 125$ $3^4 = 81$ $4^{-2} = \frac{1}{16}$ $p^r = q$

Example: Evaluate each of the following w/o a calculator

$\log_5 25$ $\log_2 32$ $\log_3 \frac{1}{9}$

$\log_5 5^6$ $\log_3 243$ $\log_5 1$

Section 9-2: Logarithmic Functions

Objectives: To graph logarithmic functions
To solve logarithmic equations

Warm-up: If $f(x) = 2x + 1$, find $f^{-1}(x)$. Graph both equations.

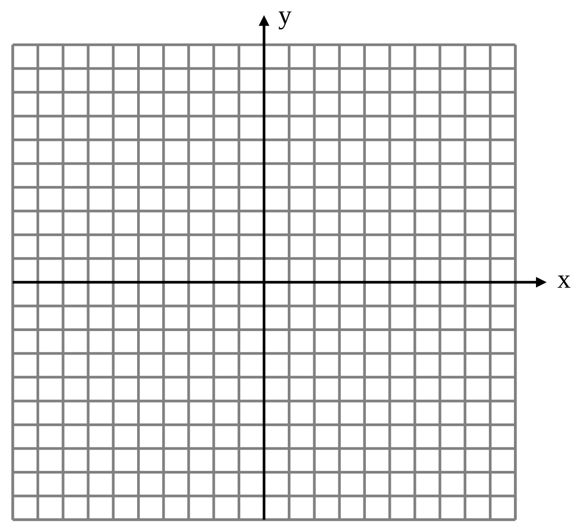
Graphing logarithmic functions

Exponential functions and logarithms are inverse functions. Recall that the inverse of a function is found by interchanging y and x.

If $y = 2^x$ then the inverse function is given by $x = 2^y$. To solve this equation for y, we'll use the definition of a logarithm: if $x = 2^y$ then $y = \log_2 x$.

$f(x) = 2^x$	
x	y

$f^{-1}(x) = \log_2 x$	
x	y



$y = 2^x$
Domain:
Range:
Vertical Asymptote:
Horizontal Asymptote:
y-intercept:

$y = \log_2 x$
Domain:
Range:
Vertical Asymptote:
Horizontal Asymptote:
x-intercept:

Solving simple logarithm equations

Logarithmic equations can frequently be solved by changing the log equation into the exponential form.

Solve for x:

$\log_7 49 = x$	$\log_5 x = 3$	$\log_3 x = -2$	$\log_x 64 = 2$
$\log_x 5 = 1$	$\log_4 64 = x + 2$	$\log_3(2x - 1) = 2$	$\log_6(x^2 + 11) = 2$

Other logarithmic equations can be solved using the Property of Equality for Logarithms: $\log_b x = \log_b y$ if and only if x = y.

Solve each equation for x. Remember, we can only take the log of positive numbers, so be sure to check the domain for values of x which would result in an undefined logarithm.

$$\log_4 x^2 = \log_4(4x - 3)$$

$$\log_5\left(x^2 - 2\right) = \log_5 x$$

Section 9-3: Properties of Logarithms

Objectives: To simplify and evaluate expressions using properties of logarithms

Warm-Up: Evaluate the following logarithms:

$\log_3 27$ $\log_3 3$ $\log_3 81$

$\log_2 4$ $\log_2 8$ $\log_2 32$

Are the following statements true?

$\log_3 27 + \log_3 3 = \log_3 81$ How do we get 81 from 27 & 3?

$\log_2 32 - \log_2 4 = \log_2 8$ How do we get 8 from 32 and 4?

Product Property of Logarithms

For all positive numbers m, n, and b, $b \neq 1$

$$\log_b m + \log_b n = \log_b (mn)$$

Quotient Property of Logarithms

For all positive numbers m, n and b, $b \neq 1$,

$$\log_b m - \log_b n = \log_b \left(\frac{m}{n}\right)$$

Example: Rewrite each of the following as one logarithm.

$$\log_3 6 + \log_3 5 =$$

$$\log_3 36 - \log_3 18 =$$

$$\log_7 x^3 - \log_7 x =$$

From the Product Property of Logs, we can rewrite

$$\log_7 5^3 = \log_7 5 \bullet 5 \bullet 5 = \log_7 5 + \log_7 5 + \log_7 5 = 3 \log_7 5$$

This suggests the Power Property of Logarithms:

For all positive numbers m, n and b, b $\neq$ 1, $\log_b m^p = p \log_b m$

Example: Rewrite each of the following as a single logarithm

$$2\log_3 10 - 2\log_3 5$$

$$2\log x + \frac{1}{2}\log y$$

Example: Write each of the following in terms of log M and log N

$$\log M^4 N$$

$$\log \frac{M^3}{\sqrt{N}}$$

Example: Find $\log_2 48$ if $\log_2 3 = 1.585$

Example: Find $\log_5 250$ if $\log_5 2 = .4307$

Example: If $x = \log_3 2$ and $y = \log_3 5$, find $\log_3 100$ in terms of x and y

Section 9-3: Properties of Logarithms, continued

Objectives: To use the properties of logarithms to solve more complicated equations

Warm-up: Review solving simple logarithmic equations

$\log_9 x = 2$	$\log_5 (x+25) = 3$	$\log_2 \frac{x^4}{3} = \log_2 27$

To solve more complicated equations containing logarithms, you will need to simplify each side of the equation so that there is no more than one logarithm on each side.

Example: Solve each of the following. Remember to check the domain.

$$2\log_8 x = \log_8 81$$

$$\log_6 9 - \log_6 x = 2$$

$$\log_7 27 + \log_7 3 = 4\log_7 x$$

$$\log(x^2+3)-\log(x-1)=\log 7$$

$$\log_4 x+\log_4(x-6)=2$$

Example : The amount of energy, E, in ergs, released by an earthquake is related to its Richter scale magnitude, M, by the equation

$\log_{10} E = 11.8 + 1.5\,M$

a. The Haiti earthquake in 2010 had a magnitude of 7.0. How much energy did this earthquake release?

b. The September 2017 earthquake in Mexico City measured 8.2 on the Richter scale. How much energy did the earthquake release?

c. The Mexico earthquake released how many times more energy than the Haiti quake?

Section 9-4: Common Logarithms

Objectives: To solve exponential equations using common logs
To use common logs to evaluate logarithms with different bases

Warm-up: Find a common base to solve for x.

$4^x = 64$ $10^x = .001$ $8^x = 16^{3x+2}$

<u>The Common Log</u>
As we have mentioned before, the "log" button on your calculator refers to log base 10 or $\log_{10} x$. Log base 10 is called the common log. The common log tells you the power of 10 in a number.

Evaluate each common log <u>without</u> using your calculator:
log 1000 log 100 log 10 log 1 log 0.1

Between what two integers is
log 546 log 54.6 log 5.46 log 0.546

Use your calculator to find
log 546 log 54.6 log 5.46 log 0.546

Can we find a common base to solve an equation like $7^x = 55$?

To solve this equation, we will need to take the common log of both sides in order to "undo" the exponent.

Solve $7^x = 55$	Take the (common) log of both sides
$\log 7^x = \log 55$	Use the Power Property of logarithms
$x \log 7 = \log 55$	Solve the equation for x
$x = \frac{\log 55}{\log 7} = \frac{1.740}{.845}$	Use a calculator to evaluate the common logs

$x = 2.059$

Solve the following:

$4^x = 127$

$6 + 3^{2x} = 450$

$7^{x+5} = 9^{2x+3}$

Suppose we want to evaluate $x = \log_5 40$.

Change to exponential form:

Now take the common log of both sides:

And solve for x:

This is called the Change of Base Formula and it lets you evaluate logs other than base 10. In general,

$$\log_a n = \frac{\log_{10} n}{\log_{10} a}$$

Use common logs and your calculator to evaluate the following:

$\log_3 180$

$\log_7 5$

Problem Solving Using Logarithms

Objectives: To learn how to calculate compound interest and to calculate the amount of time required to increase the value of an investment.

The formula for simple interest is $i = \text{Pr}\,t$, where P is the principal invested, r is the interest rate as a decimal, and t is the length of time. In this formula, interest is only calculated at the end of the time period. The amount of money you would have at the end of the period would be your principal plus the interest you earned:

$$A = P + \text{Pr}\,t = P(1 + rt)$$

Example: How much would a \$1000 investment be worth after 4 years with 5% simple interest?

Although it's very easy to use, banks do not calculate interest by this formula. Most banks use a formula that allows interest to compound a certain number of times each year. When interest is compounded, the interest earned starts earning its own interest.

Example: How much would a \$1000 investment be worth after each of 4 years with 5% interest compounded annually?

Write a formula to find the value after t years.

What would happen if we could compound interest more often?

The compound interest formula is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A = the amount of money in the account at any time
P = the principal, or original amount of money
r = the interest rate (as a decimal)
n = the # of times interest is compounded each year
t = number of years the money is invested

Example: How much would a $10,000 investment be worth after 10 years in an account paying 6% interest, if the interest is

a) simple

b) compounded annually

c) compounded semi-annually

d) quarterly

e) monthly

f) daily

Example: How long will it take a $1000 investment to double in value if it is invested in an account paying 5 ½% compounded quarterly?

Does the amount of the initial investment affect the doubling time?

Example: Suppose you bought a baseball card today for $150 and its value tripled in 8 years. What would be its annual percentage of increase in value?

Rates of depreciation are always calculated yearly, so n = 1. For depreciation, r is negative, so the formula is

$$A = P\left(1 - r\right)^{t}$$

Example: Suppose you bought a car that costs $25,000. If the value of your investment depreciates at a rate of 15% per year, what will be its value in 4 years?

Example: A coat that you really like costs $198. During a clearance sale, its price will decrease by 10% each week. How many weeks will it take until the coat costs half of its original price?

Section 9-5: Base *e* and Natural Logarithms

Objectives: To calculate continuous compound interest
To solve exponential equations using natural logarithms

Let's invest $1 at an interest rate of 100% for 1 year:

Annually:

Semiannually:

Quarterly:

Monthly:

Weekly:

Daily:

Hourly:

Every minute:

Every second:

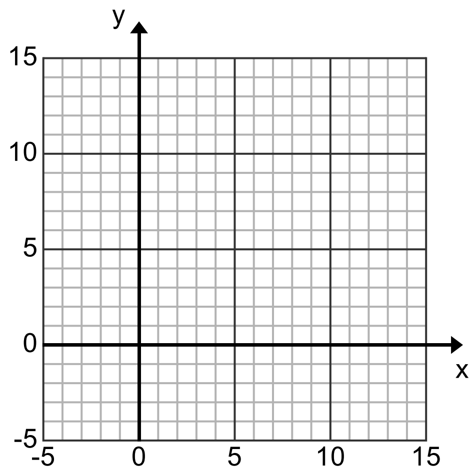
What do you notice? Do you think our investment will ever be worth $3?

Mathematicians call the value that our investment is approaching *e*. It is an irrational number (what does that mean?) Just like π is a very important irrational number in geometry, *e* is important for exponential growth. The approximate value of *e* is $e = 2.718$

Keep in mind that *e* is just a number and you can use it like any other number. Use your calculator to evaluate

e^{1} e^{2} e^{5} e^{0} e^{-1}

Use your calculator to graph $y = e^x$. Compare it to $y = 2^x$ and $y = 3^x$



Just as logarithms base 10 have a special name (what is it?), logarithms base *e* have a special name, the natural logarithm.

$$\log_e x = \ln x$$

Use your graphing calculator to examine the graph of $y = \ln x$
What are the Domain and Range of the graph?

Estimate the value of each natural log. Then use your calculator:

$\ln 9$ $\ln 1$ $\ln \frac{1}{3}$ $\ln e^2$

All of the properties of exponential and logarithmic functions apply to $y = e^x$ and $y = \ln x$.

Solve each of the logarithmic equations by rewriting as exponential equations.

$\ln 3x = 2$ $\qquad$ $\ln(2x-3) = 2.5$ $\qquad$ $\ln x + \ln 3x = 12$

Solve the following equations by taking the natural log of both sides:

$e^x = 7$ $\qquad$ $2e^x - 1 = 13$ $\qquad$ $3e^{-2x} + 4 = 10$

Interest Compounded Continuously

As we've seen, when interest is compounded more and more frequently, it will eventually reach a limit. The formula for finding the amount A earned when interest is compounded continuously is

$$A = Pe^{rt}$$

where P is the principal (the amount of money put into the account initially) and r is the annual interest rate as a decimal.

Example: How much money will you have if you invest $1 at 100% compounded continuously for 1 year?

Example: Suppose you deposit $700 into an account paying 3.5% annual interest, compounded continuously.

How much money is in the account after 3 years?

How long will it take for the amount of money to double?

How long will it take for the balance in the account to reach $2000?

Review: Solving Equations containing Exponents and Logarithms

Examples	Key Feature	What to do	Solution
$(x+1)^4 = 625$ $\sqrt[3]{x} = 4$	Variable is in the base of a polynomial equation.	Take the n^{th} root (or n^{th} power) of both sides	
$4^x = 256$ $3^{2x+1} = 81$ $32^{5p+2} = 16^{5p}$	Variable is in the exponent of an exponential equation, and both sides have a common base.	Rewrite both sides in the common base and set exponents equal to each other.	
$4^x = 250$ $e^{x+3} = 27$	Variable is in the exponent, but there is no common base.	Take the log (or ln) of both sides. Apply power rule.	
$\log_5 x = 2$ $\log_3(2x - 1) = 2$ $\ln 5x = 4$	Logarithm on one side of the equation	Rewrite equation in the exponential form	
$\log_5 (x^2 - 2) = \log_5 x$ $3\log x - \log 4 = \log 16$ $\ln x + \ln 3x = \ln 12$	Logs with the same base on both sides of the equation. May need to combine logs on one side.	Undo the log function with the equality property ($\log x = \log y$ iff $x = y$)	